Can We Uniquely Reconstruct a Function from Measurements in Which Input Is Known with Interval Uncertainty?

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Formulation of the problem. In many practical situations, we know that the value of the quantity y is uniquely determined by the value of a related quantity x. In mathematical terms, this means that y is a function of x: $y = f(x)$. However, we often do not know the exact expression for this dependence. In such situations, to find this dependence, we set up a value x and measure the corresponding value y. For example, to find possible deviations from Ohm's law, we set up different voltages x and measure the corresponding current y.

In the ideal case, when we can set up x with absolute accuracy and measure y with absolute accuracy, we get many pairs $(x, f(x))$. So, after using values x which are densely distributed in the given range, we can practically uniquely determine all the values of the desired function $f(x)$.

In practice, often, while we can measure y accurately, we can only measure x with some accuracy δ. So, based on the measurement result $\widetilde{x}$, we can only guarantee that the actual value x is in the interval $[\widetilde{x}-\delta, \widetilde{x}+\delta]$ of width 2δ. In principle, for any x from this interval, we can have measurement result $\widetilde{x}$, and if we repeat measurement many times, we will encounter situations covering many such x's. For each such x, we measure the value $f(x)$. So, after all these measurements, we get, in effect, the set

$$[\underline{y}(\widetilde{x}), \overline{y}(\widetilde{x})] \stackrel{\text{def}}{=} \{f(x) : x \in [\widetilde{x}-\delta, \widetilde{x}+\delta]\}$$

of all possible values y corresponding to different values $x \in [\widetilde{x}-\delta, \widetilde{x}+\delta]$.

A natural question is: for a fixed accuracy δ, can we uniquely reconstruct the function $f(x)$ based on these ranges? And if we cannot always reconstruct the desired function uniquely, when *can* we do it?

Sometimes we can: case of monotonicity. When the function $f(x)$ is (non-strictly) increasing, then $\underline{y}(\widetilde{x}) = f(\widetilde{x}-\delta)$ and $\overline{y}(\widetilde{x}) = f(\widetilde{x}+\delta)$. Thus, by considering appropriate values $\widetilde{x}$, we can find the value $f(x)$ for all x.

Similarly, when the function $f(x)$ is non-strictly decreasing: in this case, $\underline{y}(\widetilde{x}) = f(\widetilde{x}+\delta)$ and $\overline{y}(\widetilde{x}) = f(\widetilde{x}-\delta)$. So here also, by considering appropriate values $\widetilde{x}$, we can find the value $f(x)$ for all x.

A more general case is when the range of the function consists of several intervals $[\underline{X}, \overline{X}]$ on each of which $f(x)$ is either increasing or decreasing, and the width of each such interval is at least 4δ.

Then, all the values from $f(x)$ for $x \in [\underline{X}, \overline{X}-2\delta]$ can be obtained as $f(\widetilde{x}-\delta)$ for $\widetilde{x} = x+\delta$. For this $\widetilde{x}$, we have $[\widetilde{x}-\delta, \widetilde{x}+\delta] \subseteq [\underline{X}, \overline{X}]$, so $f(x)$ is monotonic on $[\widetilde{x}-\delta, \widetilde{x}+\delta]$ and $f(\widetilde{x}-\delta)$ can be computed.

Similarly, all the values from $f(x)$ for $x \in [\underline{X}+2\delta, \overline{X}]$ can be obtained as $f(\widetilde{x}+\delta)$ for $\widetilde{x} = x-\delta$. For this $\widetilde{x}$, we have $[\widetilde{x}-\delta, \widetilde{x}+\delta] \subseteq [\underline{X}, \overline{X}]$, so $f(x)$ is monotonic on $[\widetilde{x}-\delta, \widetilde{x}+\delta]$ and $f(\widetilde{x}+\delta)$ can be computed. Since $\overline{X}-\underline{X} \geq 4\delta$, we have $\underline{X}+2\delta \leq \overline{X}-2\delta]$ and thus, every point from the range $[\underline{X}, \overline{X}]$ is thus covered.

Sometimes, we cannot. If the range of monotonicity is too narrow, we may not be able to reconstruct the function uniquely. For example, if $f(x) = \sin(x)$, then for $\delta = \pi$ each interval $[\widetilde{x}-\delta, \widetilde{x}+\delta]$ contains the full period of the function. So, we have $[\underline{y}(\widetilde{x}), \overline{y}(\widetilde{x})] = \{f(x) : x \in [\widetilde{x}-\delta, \widetilde{x}+\delta]\} = [-1, 1]$ for all $\widetilde{x}$. The same measurements will happen if $f(x) = \cos(x)$, so the function cannot be uniquely determined.

Remaining open question. It is not clear which functions can be uniquely reconstructed and which cannot. Our hypothesis is that the $\geq 4\delta$ width of all monotonicity ranges is not only sufficient but also necessary for reconstruction uniqueness.