How Can There Be Objective Imprecise Probability

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Subjective imprecise probabilities. Often, we only have partial information about the corresponding probabilities. This is known as *imprecise probabilities.*

Objective imprecise probabilities: what are they? To understand this, let us recall what is probability from the practical viewpoint. In practice, probability p means, in effect, a frequency. We have a large number N of similar events (e.g., flipping a coin). These can be similar events occurring at different location and/or at different times. Probability p of a certain outcome means that this outcome is observed in $\approx p \cdot N$ cases.

An ideal case is when the event settings are absolutely identical. For example, we have a large set of identical atoms of a radioactive element, and we observe how many of them emit radiation during a given period of time.

In the usual quantum description, all the atoms are identical. However, the true quantum description is more complex. In the first approximation – traditional quantum mechanics – particle locations and velocities are only known with probabilities, they can fluctuate around their classical values, but the forces between particles are described by the usual formulas, e.g., Coulomb law $F = -c \cdot \dfrac{q_1 \cdot q_2}{r^2}$. In secondary quantization, we take into account that the forces can also fluctuate around the classical values. In other words, the fields – that describe these forces – are also quantum objects whose values are only known with some probabilities.

In general, no matter what kind of events we consider, these events are not identical. There are always quantum fluctuations because of which, for each event, the probability p_i is slightly different from p. Here, the values p_i are randomly fluctuating around the classical value p. In other words, here, we have objective imprecise probabilities. But what does this mean in terms of observations? Can we experimentally detect the difference between precise and imprecise probabilities? To answer this question, let us recall what randomness means in terms of observations.

What does randomness mean in terms of observations: reminder. Randomness means more than frequency. For example, according to Central Limit Theorem, differences between frequency and probability should be normally distributed. The general idea is that if a sequence is random, it must satisfy all the probability laws. A probability law is something that happens with probability 1. In mathematical terms, it is a set of probability measure 1 – so that its complement has measure 0. So, a sequence is random if it does not belong to any definable set of probability measure 0, or, equivalently, it does not belong to the union of all definable sets of measure 0. This is Kolmogorov's definition of a random sequence.

So can we experimentally detect the difference between precise and imprecise probabilities? We are interested in a sequence of events. Let $n_i = 1$ is the selected outcome occurred and $n_i = 0$ if it did not. We compare two cases: precise case when each n_i occurs with probability p, and imprecise case when each n_i occurs with probability p_i. Here, we select some distribution on the set of all probabilities with mean p, and take, as p_i, a random sequence of independent values corresponding to these probabilities.

For both sequences, we can compare moments, i.e., averages over i from 1 to N of products $n_i^{a_1} \cdot n_{i+i_1}^{a_2} \cdot \ldots$. For example, mean is the average of n_i, covariance with next neighbor depends on $n_i \cdot n_{i+1}$, etc. Our *first result* is that for both sequences, each moment tends to the same limit: e.g., the mean tends to p. However, our *second result* is that for no sequence can be random with respect to both precise and imprecise distributions. This means that there are probability laws that are only true for "imprecise" sequences. So, it *is* possible to experimentally detect the difference!