Why non-invertible symmetries

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Symmetries are important. One of the main objectives of science and engineering is to predict the future (this is the science part) – and to use this prediction ability to perform actions and/or design gadgets that will make the future better (this is the engineering part). The main idea behind prediction is that we expect the outcome to be the same as in similar situations in the past. Here, "similar" means that there may be some differences between the current situation and the past situations, some changes (= transformations), but these changes do not affect the desired behavior. For example, in the past, the water boiled at 100°. We can move around, we can change the color of the teapot, we can rotate it – the result will be the same: once the temperature reaches 100°, water starts boiling. In physics, such transformations that do not affect the analyzed phenomenon are known as *symmetries*.

Symmetries are usually invertible. Most physical symmetries are invertible: if we move 100 m in one direction, we can always move back and thus get back to the original state. If we turn 90 degrees to the right, we can always turn back. If we replace all positive electric changes with negative ones and vice versa – a transformation that preserves all electromagnetic interactions – then we can replace them back and thus, get back to the original state.

Non-invertible symmetries emerge. Recently, it was discovered that in quantum systems, an important role is played by non-invertible symmetries, in which two different states may be transformed into the same state. For example, in addition to an invertible symmetry S that swaps $|0\rangle$ and $|1\rangle$ states of a qubit – and that applies a similar swap to all the qubits in a multi-qubit system – it makes sense to consider a transformation $T : x \mapsto \dfrac{1}{\sqrt{2}} \cdot x + \dfrac{1}{\sqrt{2}} \cdot S(x)$ that also preserves some important properties of a physical system. For example, for the 2-qubit state $x = |01\rangle$, the resulting state is $T(x) = \dfrac{1}{\sqrt{2}} \cdot |01\rangle + \dfrac{1}{\sqrt{2}} \cdot |10\rangle$.

Formulation of the problem: non-invertible symmetries help, but why? In several applications, non-invertible symmetries turnn out to be more helpful than invertible ones. A natural question is: why?

Our explanation. Our explanation is based on calculus. According to calculus, the maximum of a function on a domain is attained either in D's interior or on its border; if the maximum is attained in the interior, then at the corresponding point, all partial derivatives of the maximized function should be equal to 0 – i.e., in mathematical terms, this point should be a stationary point of this function. Many functions have only a few stationary points. So, if the domain is small, the probability that this domain contains one of the stationary points is also small. Thus, in most cases, the maximum is attained on the border of the domain.

In our case, the domain D is the set all invertible transformations T that preserve certain properties, and the function that we trying to maximize is some characteristic of usefulness of the transformation T in analyzing the corresponding physical phenomena. We can therefore conclude that in many cases, the most effective transformations are the ones located on the border of D. Any invertible linear transformation is in the interior of D, since a small perturbation of an invertible matrix keeps it invertible – thus, the border of D consists of non-invertible transformations.