How to take negative information into account?

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Formulation of the problem. In many practical situations, we rely on expert estimates. When several experts provide estimates $x_1, \ldots, x_n$ for the quantity of interest, then a natural idea is to use the arithmetic mean $x = (x_1 + \ldots + x_n)/n$ in our decision making. But what if some experts provide *negative* information – e.g., saying that the actual value is *not* close to some value y_j? How can we take this information into account? In this talk, we use decision theory approach to answer this question.

What if there is no negative information. To deal with the problem, let us first consider the case when there is no negative information, i.e., when all experts submit some estimates x_i. According to decision theory, we should select an alternative with the largest value of expected utility. For our situation, this means that we select the value x that maximized the expression $p_1 \cdot u_1(x) + \ldots + p_n \cdot u_n(x)$, where p_i is the probability that the i-th expert is correct, and $u_i(x)$ is the utility according to this expert. In situations when we have no reason to assume that some experts are more skilled than others, it is reasonable to consider them equally probable to be correct, i.e., $p_1 = \ldots = p_n$.

What is $u_i(x)$? The fact that the i-th expert provides an estimate x_i means that, according to this expert, the utility attains its maximum when $x = x_i$. Since they are experts, their estimates x_i are close to the actual value a, so the difference $x_i - a$ is small. In this case, we can safely ignore higher order terms in the Taylor expansion of $u_i(x)$ and only keep the first few terms. It is not possible to only keep linear terms, since a linear function does not have a maximum. So, we need to also take quadratic terms into account, so $u_i(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2$. Since the function $u_i(x)$ attains its maximum for $x = x_i$, this means that $u_i(x) = A_i - B_i \cdot (x - x_i)^2$ for some A_i and $B_i > 0$. Again, since there is no reason to assume that the values A_i and B_i are different, it makes sense to assume that they are the same for all experts, i.e., that $A_i = A$ and $B_i = B$ for some A and B. In this case, the expected utility takes the form $A - (B/n) \cdot \sum (x - x_i)^2$, so maximizing it is equivalent to minimizing the sum of the squares $\sum (x - x_i)^2$. And it is well known that minimizing this expression leads to the arithmetic mean.

What if in addition n positive experts, m experts provide negative information? The opinion that the actual value is *not* close to y_j means that the corresponding utility function attains its *smallest* value when $x = y_j$. In this case, a similar argument leads to the expression $u_j(x) = C_j + D_j \cdot (x - y_j)^2$ for some C_j and $D_j > 0$. It still makes sense to assume that the values C_j and D_j are the same for all negative experts: $C_j = C$ and $D_j = D$ for some C and D.

Thus, the expected utility takes the form $\text{const} - (B/(n+m)) \cdot \sum (x - x_i)^2 + (D/(n+m)) \cdot \sum (x - y_j)^2$. If we subtract the constant and divide the maximized function by $B/(n+m)$, we conclude that maximizing utility is equivalent to minimizing the sum $\sum (x - x_i)^2 - k \cdot \sum (x - y_j)^2$, where $k = D/B$. Differentiating this expression with respect to x and equating the derivative to 0, we get: $x = (1/(n - k \cdot m)) \cdot (x_1 + \ldots + x_n - k \cdot (y_1 + \ldots + y_m))$.