Introducing Metric Spaces As Indexed Families of Graphs

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1 Abstract

At a glance, a metric space is a set equipped with a notion of distance between its elements. Metric spaces are a general concept with applications throughout mathematics, including analysis, geometry, and algebra. Formally, a *metric space* is a pair (S, d) consisting of a set S and a metric function d that separates points, is symmetric, and satisfies the triangle inequality. Familiar examples include $\mathbb{R}^2$ equipped with the Euclidean, Manhattan, or maximum (Chebyshev) metric. Additionally, a metric can be thought of as an indexed family of graphs, which are its level sets. The domain of a metric is a set of pairs, and the level sets are sets of pairs which we may think of as edges specifying a graph. As an example, we may specify a metric on $\mathbb{R}^2$ by providing an indexed family of graphs on $\mathbb{R}^2$: we have a graph that connects points when they have distance 1 apart, the graph that connects points when they have distance 3/2 apart, and so on. Tracking these levels sets is a way to study the underlying metric space. These basic notions serve as a guide for further refinements and for defining a quantum analogue of a metric.

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