Gaudí arches revisited

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Formulation of the problem. Buildings designed by the famous architect Antoni Gaudí have many arches. These arches are either parabolic $y = a - bx^2$, or of weighted catenary shape $y = b \cdot (e^{x/a} + e^{-x/a})$ [1]. How can we explain this?

Analysis of the problem and the resulting explanation. We want to find the formula $y = f(x)$ that describes the geometry of an arch.

To make the formulas simpler, let us take the top of the arch as the starting point $(0, 0)$ of the coordinate system (so that $f(0) = 0$), and let the vertical coordinate y go down from there, so that all the values y corresponding to the arch should be non-negative.

The arch is symmetric with respect to the change from x to $-x$, so we should have $f(x) = f(-x)$.

We also want to make sure that the function $f(x)$ describes an (inverted) arch, i.e., that $f(x)$ is decreasing for $x < 0$ and increasing for $x > 0$.

In science, a natural way to describe each dependence $y = f(x)$ is by a differential equation, usually of the first or of the second order. Due to the symmetry of the arch, this equation should not change when we change from x to $-x$.

From this point of view, let us first consider the first order equations $y' = f(y)$. When we replace x with $-x$, $f(y)$ does not change, but the y' changes to $-y'$. So, first-order equations cannot describe a symmetric arch, we need a second-order equation $y'' = f(y, y')$ for some function $f(y, y')$.

In general, in physics, when do not know a function that describes the dependence between physical quantities, we take into account that functions can usually be expanded into Taylor series – and thus, can be approximated by the first few terms in this expansion. In our case, we can consider the Taylor expansion of the unknown function $f(y, y')$:

$$f(y, y') = a_0 + a_{10} \cdot y + a_{01} \cdot y' + a_{20} \cdot y^2 + a_{11} \cdot y \cdot y' + a_{02} \cdot (y')^2 + \ldots$$

If we only take into account the 0-th order terms in the expansion of the function $f(y, y')$, we get the equation $y'' = a_0$. Its general solution is a *parabola*. If we also take into account linear terms, we get the equation $y'' = a_0 + a_{10} \cdot y + a_{01} \cdot y'$. For this equation to be invariant under $x \to -x$, we need to take $a_{01} = 0$, so we get $y'' = a_0 + a_{10} \cdot y$. For $a_{10} < 0$, a general solution is a sinusoid that does not satisfy the monotonicity condition. So, $a_{10} > 0$. In this case, a general symmetric solution is indeed of *weighted catenary* shape.

So, we have explained both shapes of Gaudí arches.

[1] G. González, A. Samper, and B. Herrera, "Classification by type of the arches in Gaudí's Palau Güell", *Nexus Network Journal: Architecture and Mathematics*, 2018, Vol. 20, pp. 173–186,
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